

On completeness in a non-Archimedean setting, via firm reflections

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Abstract

We develop a completion theory for (general) non-Archimedean spaces based on the theory on "a categorical concept of completion of objects" as introduced by G.C.L. Brümmer and E. Giuli in [7]. Our context is the construct \mathbf{NA}_0 of all Hausdorff non-Archimedean spaces and uniformly continuous maps and \mathcal{V} is the class of all epimorphic embeddings in \mathbf{NA}_0 .

We determine the class $\mathbf{Inj} \mathcal{V}$ of all \mathcal{V} -injective objects and we present an internal characterization as "complete objects". The basic tool for this characterization is a notion of small collections that in some sense preserve the inclusion order on the non-Archimedean structure. We prove that the full subconstruct \mathbf{CNA}_0 consisting of all complete objects forms a firmly \mathcal{V} -reflective subcategory. This means that every object X in \mathbf{NA}_0 has a completion which is a \mathcal{V} -reflection $r_X : X \rightarrow RX$ into the full subconstruct \mathbf{CNA}_0 of "complete spaces". Moreover this completion is unique (up to isomorphism) in the sense that, considering $L(\mathbf{CNA}_0)$, the class of all those morphisms $u : X \rightarrow Y$ for which $Ru : RX \rightarrow RY$ is an isomorphism, one has that \mathcal{V} is contained in $L(\mathbf{CNA}_0)$. In fact one even has $\mathcal{V} = L(\mathbf{CNA}_0)$.

Finally we apply our constructions to the classical case of Hausdorff non-Archimedean uniform spaces, in that case our completion reduces to the standard one [21], [22].

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1 Introduction

Non-Archimedean uniform spaces were introduced in 1950 by Monna in [17]. They play an important role in non-Archimedean Analysis. First of all the uniformity of the scalar field \mathbb{K} , induced by a non-Archimedean valuation is itself non-Archimedean, and secondly the non-Archimedean property is preserved by uniform products and subspaces. In fact every space obtained from the scalar

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field by any initial construction, is a non-Archimedean uniform space. Moreover non-Archimedean uniformities have proven to provide the right notion of uniformizability for zero-dimensional topological spaces and they inherit all nice features from the "exemplary completion theory" for uniform spaces, the meaning of which is explained below.

Quoting B. Banaschewski from his 1955 paper [5], non-Archimedean structures "belong to the subfield of General Topology that can be described by means of equivalence relations". For Monna [17] a non-Archimedean uniformity on X is a uniformity generated by a filter base of equivalence relations on X . In our present setting, we have to allow for somewhat more general non-Archimedean structures. Basically we will work with stacks of equivalence relations (see Definition 2.1). The main reason for passing to the more general setting, is to include some mathematical structures used in the representation theory of certain systems. For references see for instance G. Aumann's work on contact relations or more recent work of B. Ganter and R. Wille on formal concept analysis [4], [13]. These models combine topological and lattice-theoretical ideas, and often make use of so called Birkhoff closures [6], which do not necessarily satisfy the finite additivity of the usual Kuratowski (i.e. topological) closure. This is in particular the case with models for physical systems, built on a well defined "lattice of properties of the system" as developed by Aerts in [2] and by Moore in [18]. There is a natural Birkhoff closure corresponding to this lattice and recently in [3] it was shown that in this correspondence, the lattice of "classical physical properties" gives rise to a zero-dimensional Birkhoff closure space. The non-Archimedean structures considered in this paper provide the right notion of uniformizability for these zero-dimensional closures. Moreover, as in the classical case, they are also stable for initial constructions.

Our main concern in this note will be with the completion theory for these (general) non-Archimedean spaces. We will apply a "categorical concept of completion of objects" as developed by G.C.L. Brümmer and E. Giuli in [7] to the setting of non-Archimedean spaces. These authors started from the "exemplary" behavior of the usual completion in the category $\mathbf{X} = \mathbf{UNIF}_0$ of Hausdorff uniform spaces with uniformly continuous maps. If \mathcal{V} is the class of all dense embeddings in \mathbf{X} then, as is well known, every object X in \mathbf{X} has a completion which is a \mathcal{V} -reflection $r_X : X \rightarrow RX$ into the full subconstruct \mathbf{R} of "complete spaces". Moreover this completion is unique (up to isomorphism) in the sense that, considering $L(\mathbf{R})$, the class of all those morphisms $u : X \rightarrow Y$ for which $Ru : RX \rightarrow RY$ is an isomorphism, one has that \mathcal{V} is contained in $L(\mathbf{R})$. In the case of $\mathbf{X} = \mathbf{UNIF}_0$ one even has $\mathcal{V} = L(\mathbf{R})$. To describe this exemplary behavior of the subcategory \mathbf{R} of complete objects in \mathbf{X} the authors of [7] used the terminology " \mathbf{R} is firmly \mathcal{V} -reflective in \mathbf{X} ". The title of our paper refers to this terminology.

Our context will be the construct \mathbf{NA}_0 of all Hausdorff non-Archimedean spaces and uniformly continuous maps, as introduced in Definitions 2.1 and 2.2. First we determine the class \mathcal{V} of all epimorphic embeddings in \mathbf{NA}_0 . Contrary to the classical case, the epimorphisms are no longer the dense maps described by the underlying closure of the space. In Theorem 2.6 we characterize the

derived closure operator needed to do the job. From the general results in [7] we know that if \mathbf{NA}_0 admits a firmly \mathcal{V} -reflective subconstruct \mathbf{R} then \mathbf{R} has to coincide with the class $\mathbf{Inj} \mathcal{V}$ of all \mathcal{V} -injective objects. In Theorem 3.7 we give an internal characterization of these injective objects. The basic tool for this characterization is the notion of small collections (i.e. collections of subsets containing arbitrarily small sets) that in some sense preserve the inclusion order between the equivalence relations which determine the structure. These are used to define our "complete objects" and we prove that the full subconstruct consisting of all complete objects indeed forms a firmly \mathcal{V} -reflective subcategory. We add an explicit description of the "completion". Finally we apply our constructions to the classical situation of Hausdorff non-Archimedean uniform spaces, in which case our completion reduces to the standard one [21], [22].

For categorical terminology we refer to books such as [1], [15] or [20] and for terminology and results on closure operators useful references are the original papers [10] and [11] or the recent book [12].

2 The construct of non-Archimedean spaces

In this section we develop the context of our completion theory. We introduce the construct of Hausdorff non-Archimedean spaces and we pay particular attention to special morphisms in this construct since in the next paragraph the class of all epimorphic embeddings will play a key role.

Definition 2.1. *A non-Archimedean structure \mathcal{E} on a set X is a stack of equivalence relations on X , i.e. a collection \mathcal{E} of equivalence relations on X satisfying:*

$$E \in \mathcal{E}, E \subset E', E' \text{ equivalence relation on } X \Rightarrow E' \in \mathcal{E}$$

The couple (X, \mathcal{E}) is called a non-Archimedean space.

Obviously, instead of working with equivalence relations on X one could consider partitions of X instead. So an alternative for Definition 2.1 is to consider a set β of partitions of X satisfying:

$$\mathcal{P} \in \beta, \mathcal{P} \prec \mathcal{P}', \mathcal{P}' \text{ partition on } X \Rightarrow \mathcal{P}' \in \beta$$

where \prec is the refinement relation defined on covers $\mathcal{P} \prec \mathcal{P}'$ (\mathcal{P} refines \mathcal{P}') iff $\forall P \in \mathcal{P} : \exists P' \in \mathcal{P}' : P \subset P'$.

We will use the following notations. A non-Archimedean space will be written as \mathbf{X} and we shall use β_X and \mathcal{E}_X to refer to the corresponding structures and X for the underlying set. We will write $[x]_{\mathcal{P}}$ as well as $E[x]$ for the equivalence class of a point $x \in X$.

Remark that every partition star refines itself. Therefore every non-Archimedean space provides a base for a pre-nearness space in the sense of [20]. It is even a base for a uniform semi-nearness space as in [9]. Moreover if \mathcal{E}_X is closed under finite intersections then it forms a base for a collection of entourages of

a non-Archimedean uniform space [5], [17], [21], [22]. In this case β_X is closed under the operation \wedge given by $\mathcal{P} \wedge \mathcal{Q} = \{P \cap Q | P \in \mathcal{P}, Q \in \mathcal{Q}\}$ and it generates a uniform nearness space in the sense of [14].

Definition 2.2. *A function $f : \mathbf{X} \rightarrow \mathbf{Y}$ between non-Archimedean spaces is called uniformly continuous if*

$$\forall E \in \mathcal{E}_Y : (f \times f)^{-1}(E) \in \mathcal{E}_X$$

In terms of partitions this is obviously equivalent to

$$\forall \mathcal{P} \in \beta_Y : f^{-1}(\mathcal{P}) \in \beta_X$$

The category of non-Archimedean spaces together with the uniformly continuous maps will be denoted \mathbf{NA} . It is a topological construct in the sense of [1]. This means that initial (and final) structures for arbitrary class indexed sources (and sinks) can be formed in \mathbf{NA} . In particular the objects on a fixed underlying set X , form a complete lattice with largest element the discrete object \mathbf{D}_X and smallest one the indiscrete object \mathbf{I}_X .

Given a source $(f_i : X \rightarrow \mathbf{X}_i)_{i \in I}$ in \mathbf{NA} then the initial structure \mathcal{E} is given by $\{(f_i \times f_i)^{-1}(E_i) | i \in I, E_i \in \mathcal{E}_{X_i}\}$.

\mathbf{NA}_0 is the subconstruct consisting of the T_0 objects of \mathbf{NA} . Applying the usual definition [16] we say that \mathbf{X} is a T_0 object if and only if every uniformly continuous map from the indiscrete object $\mathbf{I}_{\{0,1\}}$ to \mathbf{X} is constant. This equivalently means that for any two different points x and y in X there is an equivalence relation $E \in \mathcal{E}_X$ such that $E[x] \neq E[y]$.

In view of this separation condition the objects in \mathbf{NA}_0 will be called Hausdorff non-Archimedean spaces. From the results of Marny in [16] it follows that \mathbf{NA}_0 is an extremally epireflective subconstruct of \mathbf{NA} and as such it is initially structured in the sense of [19], [20]. In particular \mathbf{NA}_0 is complete, cocomplete and well powered, it is an (epi - extremal mono) category and an (extremal epi - mono) category [15]. Also from the general setting [19] it follows that the monomorphisms in \mathbf{NA}_0 are exactly the injective uniformly continuous maps and a morphism in \mathbf{NA}_0 is an extremal epimorphism if and only if it is a regular epimorphism if and only if it is surjective and final.

In order to describe the epimorphisms and the extremal monomorphisms in \mathbf{NA}_0 the following result on cogenerators in \mathbf{NA}_0 is very useful.

Proposition 2.3. *Let \mathbf{X} be a Hausdorff non-Archimedean space. We have that*

$$i : \mathbf{X} \rightarrow \prod_{\mathcal{P} \in \beta_X} \mathbf{D}_{\mathcal{P}} : x \mapsto ([x]_{\mathcal{P}})_{\mathcal{P} \in \beta_X}$$

is an embedding (i.e. injective and initial).

Proof. Consider the source:

$$(i_{\mathcal{P}} : X \rightarrow \mathbf{D}_{\mathcal{P}} : x \mapsto [x]_{\mathcal{P}})_{\mathcal{P} \in \beta_X}$$

Since $(i_{\mathcal{P}} \times i_{\mathcal{P}})^{-1}(\Delta_{\mathcal{P}}) = E_{\mathcal{P}}$, where $E_{\mathcal{P}}$ denotes the equivalence relation defined by \mathcal{P} , we know that β_X is the initial structure for this source. Because \mathbf{X} is Hausdorff we have that the above source is point separating. Hence

$$i : \mathbf{X} \rightarrow \prod_{\mathcal{P} \in \beta_X} \mathbf{D}_{\mathcal{P}} : x \rightarrow ([x]_{\mathcal{P}})_{\mathcal{P} \in \beta_X}$$

is an embedding. \square

This means that Hausdorff non-Archimedean spaces form exactly the epireflective hull in \mathbf{NA} of the class of all discrete objects, or using the terminology of [8], \mathbf{NA}_0 is cogenerated, with respect to all embeddings, by the class of all discrete objects, i.e. the Hausdorff non-Archimedean spaces are exactly the subspaces of a product of discrete spaces.

To simplify notations we shall write $\Pi\beta_X$ instead of $\prod_{\mathcal{P} \in \beta_X} \mathbf{D}_{\mathcal{P}}$.

As we will see next, as opposed to the classical case, the epimorphisms in \mathbf{NA}_0 can not be described as the "dense" uniformly continuous maps, with "denseness" defined by the underlying closure. We need to determine the \mathbf{NA}_0 -regular closure operator as introduced in [10], [11] and define "denseness" accordingly.

Definition 2.4. [10], [11] *Given a non-Archimedean space \mathbf{X} and a subset $M \subset X$, a point x of X is in the regular closure of M in \mathbf{X} iff for every Hausdorff non-Archimedean space \mathbf{Z} and for every pair of uniformly continuous maps $f, g : \mathbf{X} \rightarrow \mathbf{Z}$*

$$f|_M = g|_M \Rightarrow f(x) = g(x)$$

in this case we write $x \in \text{reg}_X(M)$.

Using Proposition 2.3 we immediately obtain the following equivalent description of Definition 2.4.

$x \in \text{reg}_X(M)$ iff for every discrete space \mathbf{D} and for every pair of uniformly continuous maps $f, g : \mathbf{X} \rightarrow \mathbf{D}$

$$f|_M = g|_M \Rightarrow f(x) = g(x)$$

In order to obtain an explicit description of the regular closure operator we introduce the following notation.

Definition 2.5. *Let \mathbf{X} be a non-Archimedean space and $M \subset X$. $x \in \zeta_X(M)$ iff for every two equivalence relations $E_1, E_2 \in \mathcal{E}_X$, which coincide on M , we have that $E_1[x] \cap E_2[x] \cap M \neq \emptyset$.*

Theorem 2.6. *For every non-Archimedean space \mathbf{X} and $M \subset X$:*

$$\zeta_X(M) = \text{reg}_X(M)$$

Proof. Let \mathbf{X} be a non-Archimedean space, let $x \in X$, $M \subset X$ such that $x \notin \text{reg}_X(M)$. There is a discrete object \mathbf{D} and there are uniformly continuous maps $f, g : \mathbf{X} \rightarrow \mathbf{D}$ for which $f|_M = g|_M$ and $f(x) \neq g(x)$. Consider $E_1 = (f \times f)^{-1}(\Delta_D)$, $E_2 = (g \times g)^{-1}(\Delta_D)$. Clearly E_1 and E_2 belong to \mathcal{E}_X but do not satisfy the condition in Definition 2.5

Conversely, if $x \notin \zeta_X(M)$, choose $E_1, E_2 \in \mathcal{E}_X$ such that E_1 and E_2 coincide on M and $E_1[x] \cap E_2[x] \cap M = \emptyset$. Let $C = \{E_1[m] \cap M \mid m \in M\} = \{E_2[m] \cap M \mid m \in M\}$ and $D = C \cup \{a, b\}$ where $a, b \notin C$. Write \mathbf{D} for the discrete object on D . We define the following functions:

$$f : \mathbf{X} \rightarrow \mathbf{D} : y \mapsto \begin{cases} E_1[m] \cap M & \text{if } \exists m \in M : (y, m) \in E_1 \\ a & \text{if } \forall m \in M : (y, m) \notin E_1 \end{cases}$$

and

$$g : \mathbf{X} \rightarrow \mathbf{D} : y \mapsto \begin{cases} E_2[m] \cap M & \text{if } \exists m \in M : (y, m) \in E_2 \\ b & \text{if } \forall m \in M : (y, m) \notin E_2 \end{cases}$$

Clearly f and g are uniformly continuous, they coincide on M but $f(x) \neq g(x)$. \square

It follows from [11] that

$$\zeta : \{\zeta_X : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\}_{\mathbf{X} \in |\mathbf{NA}|}$$

defines a closure operator on \mathbf{NA} .

Clearly ζ is hereditary in the sense that for a space \mathbf{Y} and a subspace \mathbf{X} in \mathbf{NA} and $M \subset X \subset Y$, we have $\zeta_X(M) = \zeta_Y(M) \cap X$.

A subset M of a non-Archimedean space \mathbf{X} is ζ -dense (ζ -closed) if $\zeta_X(M) = X$ ($\zeta_X(M) = M$). A map $f : \mathbf{X} \rightarrow \mathbf{Y}$ between non-Archimedean spaces is ζ -dense (ζ -closed) if $f(X)$ has the corresponding property with respect to Y [10], [11].

Proposition 2.7. *In \mathbf{NA}_0 we have:*

1. *The epimorphisms are exactly the ζ -dense uniformly continuous maps.*
2. *The extremal monomorphisms coincide with the regular monomorphisms and they both coincide with the ζ -closed embeddings.*

Proof.

1. This follows from Theorem 2.8 in [11] since ζ is the regular closure operator determined by \mathbf{NA}_0 .

2. Using proposition 2.6 in [11] and the fact that \mathbf{NA}_0 is extremally epireflective in \mathbf{NA} , for an \mathbf{NA}_0 morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$ the following implications hold. (i) f ζ -closed embedding \Rightarrow (ii) f regular monomorphism \Rightarrow (iii) f extremal monomorphism \Rightarrow (iv) f embedding. To see that (iii) implies that f is ζ closed, we have to use (weak) hereditariness of ζ , either by applying 6.2 in [12] or by the

following direct argument. Let $M = \zeta_Y(f(X))$ and $h : \mathbf{M} \rightarrow \mathbf{Y}$ the associated ζ -closed embedding. Then there exists a unique map g such that

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \\ & \searrow g & \nearrow h \\ & \mathbf{M} & \end{array}$$

By the (weak) hereditariness of the ζ -closure, g is ζ -dense and so it is an epimorphism in \mathbf{NA}_0 . It follows that g is an isomorphism and then f is ζ -closed.

□

From the previous characterization of the epimorphisms in \mathbf{NA}_0 it now follows that \mathbf{NA}_0 is cowell-powered.

It suffices to observe that given an ζ -dense map $f : \mathbf{X} \rightarrow \mathbf{Y}$, with \mathbf{X} fixed, there is a one to one correspondence between $\beta_{f(X)}$ and β_Y . Therefore the cardinality of Y is uniformly bounded.

3 The firm \mathcal{V} -reflective subconstruct consisting of complete objects of \mathbf{NA}_0

In this paragraph we show that \mathbf{NA}_0 admits a completion theory that is "exemplary" in the sense explained in the introduction.

Let \mathcal{V} be the class of epimorphic embeddings of \mathbf{NA}_0 , by Proposition 2.7 \mathcal{V} consists of all ζ -dense embeddings. This class \mathcal{V} satisfies the following conditions

(α) closedness under composition

(β) closedness under composition with isomorphisms on both sides

(α) and (β) are standing assumptions made in [7] and enable us to apply to \mathbf{NA}_0 the theory developed in that paper. We will prove that \mathbf{NA}_0 admits a \mathcal{V} -reflective subconstruct \mathbf{R} which is firm in the terminology of [7]. Explicitly this means that:

1. Every object \mathbf{X} has a reflection $r_X : \mathbf{X} \rightarrow R\mathbf{X}$ into \mathbf{R} such that $r_X \in \mathcal{V}$.
2. If $L(\mathbf{R})$ is the class of all morphisms $u : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathbf{NA}_0 for which $Ru : R\mathbf{X} \rightarrow R\mathbf{Y}$ is an isomorphism then $\mathcal{V} = L(\mathbf{R})$.

By proposition 1.6 in [8], in order to construct \mathbf{R} it suffices to find a class consisting of \mathcal{V} -injective objects that cogenerates \mathbf{NA}_0 , with respect to embeddings (cf. Proposition 2.3).

Recall that a Hausdorff non-Archimedean space \mathbf{B} is \mathcal{V} -injective if for each $v : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathcal{V} and $f : \mathbf{X} \rightarrow \mathbf{B}$ uniformly continuous there exists a uniformly continuous $f' : \mathbf{Y} \rightarrow \mathbf{B}$ such that $f' \circ v = f$. In this case f' is called an extension of f along v . **Inj** \mathcal{V} denotes the full subcategory of all \mathcal{V} -injective objects in \mathbf{NA}_0 .

Proposition 3.1. *Every discrete Hausdorff non-Archimedean space is \mathcal{V} -injective.*

Proof. Let $u : \mathbf{X} \rightarrow \mathbf{Y}$ be a ζ -dense embedding between two Hausdorff non-Archimedean spaces, and let $f : \mathbf{X} \rightarrow \mathbf{D}$ be a uniformly continuous function to a discrete space. By the initiality of u we have an $E \in \mathcal{E}_Y$ such that $(u \times u)^{-1}(E) = (f \times f)^{-1}(\Delta_D)$. For $y \in Y$ we choose $x_y \in X$ such that $(y, u(x_y)) \in E$. We define

$$f^* : \mathbf{Y} \rightarrow \mathbf{D} : y \mapsto f(x_y)$$

Clearly f^* is a well-defined uniformly continuous map which is an extension of f along u . \square

Combining Propositions 2.3 and 3.1 and using Theorem 1.6 in [8] and Theorem 1.4 and Proposition 1.14 in [7] we can formulate the next result:

Proposition 3.2. *The following hold:*

1. \mathbf{NA}_0 admits a unique firmly \mathcal{V} -reflective subconstruct \mathbf{R} .
2. \mathbf{R} coincides with the class $\mathbf{Inj} \mathcal{V}$.
3. \mathbf{R} coincides with the epireflective hull in \mathbf{NA}_0 of the class of all discrete spaces.

We next present an internal characterization of the objects in the firm \mathcal{V} -reflective subconstruct. In order to do this we formulate the following.

Definition 3.3. *Let \mathbf{X} be a non-Archimedean space. As usual, a choice function is a map $f : \beta_X \rightarrow \cup \beta_X$ such that for any $\mathcal{P} \in \beta_X$ one has that $f(\mathcal{P}) \in \mathcal{P}$. A choice function is order preserving iff $\mathcal{P} \prec \mathcal{Q}$ implies $f(\mathcal{P}) \subset f(\mathcal{Q})$.*

Definition 3.4. *A Hausdorff non-Archimedean space is complete iff for every order preserving choice function f there is an $x \in \cap_{\mathcal{P} \in \beta_X} f(\mathcal{P})$.*

The point x is called a limit point of f and we will say that f converges to x . Note that in this case the limit point x is unique.

The following proposition links this concept of completeness to the firmly \mathcal{V} -reflective subcategory we described before.

Proposition 3.5. *Let \mathbf{X} be a Hausdorff non-Archimedean space and consider the embedding $i : \mathbf{X} \rightarrow \Pi \beta_X$ from Proposition 2.3. We have the following equivalence.*

$$z = (z_{\mathcal{P}})_{\mathcal{P} \in \beta_X} \in \zeta_{\Pi \beta_X}(i(X)) \iff \forall \mathcal{P}, \mathcal{Q} \in \beta_X : \mathcal{P} \prec \mathcal{Q} \Rightarrow z_{\mathcal{P}} \subset z_{\mathcal{Q}}$$

Proof. First let $z \in \zeta_{\Pi \beta_X}(i(X))$ and let $\mathcal{P} \prec \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q} \in \beta_X$. For $E_{\mathcal{P}} = (pr_{\mathcal{P}} \times pr_{\mathcal{P}})^{-1}(\Delta_{\mathcal{P}})$ and $E_{\mathcal{Q}} = (pr_{\mathcal{Q}} \times pr_{\mathcal{Q}})^{-1}(\Delta_{\mathcal{Q}})$, we know that there is an $x \in X$ such that $i(x) \in E_{\mathcal{P}}[z] \cap E_{\mathcal{Q}}[z]$. Therefore $[x]_{\mathcal{P}} = z_{\mathcal{P}} \in \mathcal{P}$ and $[x]_{\mathcal{Q}} = z_{\mathcal{Q}} \in \mathcal{Q}$. Because $\mathcal{P} \prec \mathcal{Q}$ we know that there is a $Q \in \mathcal{Q}$ such that $z_{\mathcal{P}} \subset Q$. Since $z_{\mathcal{Q}} \in \mathcal{Q}$ and both Q and $z_{\mathcal{Q}}$ contain x we have that $z_{\mathcal{Q}} = Q$. Hence $z_{\mathcal{P}} \subset z_{\mathcal{Q}}$.

Conversely, choose E_1, E_2 in $\mathcal{E}_{\Pi\beta_X}$ such that both coincide on $i(X)$. Clearly one can write $E_1 = (pr_{\mathcal{P}} \times pr_{\mathcal{P}})^{-1}(E_{\mathcal{P}})$ and $E_2 = (pr_{\mathcal{Q}} \times pr_{\mathcal{Q}})^{-1}(E_{\mathcal{Q}})$ where $E_{\mathcal{P}}$ and $E_{\mathcal{Q}}$ are equivalences on \mathcal{P} and \mathcal{Q} respectively.

For the equivalence relation $E_{\mathcal{P}}$, consider classes $[x]_{\mathcal{P}}, [y]_{\mathcal{P}} \in \mathcal{P}$ of points x and y in X , we have

$$\begin{aligned} ([x]_{\mathcal{P}}, [y]_{\mathcal{P}}) \in E_{\mathcal{P}} &\iff (i(x), i(y)) \in E_1 \\ &\iff (i(x), i(y)) \in E_2 \\ &\iff ([x]_{\mathcal{Q}}, [y]_{\mathcal{Q}}) \in E_{\mathcal{Q}} \end{aligned}$$

Hence $\mathcal{R} = \{\{y \in X \mid ([x]_{\mathcal{P}}, [y]_{\mathcal{P}}) \in E_{\mathcal{P}}\} \mid x \in X\}$ is a partition of X , such that $\mathcal{P}, \mathcal{Q} \prec \mathcal{R}$. So by the hypothesis we have $z_{\mathcal{P}}, z_{\mathcal{Q}} \subset z_{\mathcal{R}}$. Thus there is an $x \in X$ for which $(i(x), z) \in E_1$ and $(i(x), z) \in E_2$. Finally we have $z \in \zeta_{\Pi\beta_X}(i(X))$. \square

Theorem 3.6. *Let \mathbf{X} be a non-Archimedean space. The following are equivalent:*

1. \mathbf{X} is complete.
2. \mathbf{X} is a ζ -closed subspace of $\Pi\beta_X$.
3. \mathbf{X} is \mathcal{V} -injective.

Proof. We prove the following implications:

$1 \Rightarrow 2$: Suppose \mathbf{X} is complete. We prove that $i(X)$ is ζ -closed in $\Pi\beta_X$. Let $z = (z_{\mathcal{P}})_{\mathcal{P} \in \beta_X} \in \zeta_{\Pi\beta_X}(i(X))$, by Proposition 3.5 we know that $f : \beta_X \rightarrow \cup \beta_X : \mathcal{P} \mapsto z_{\mathcal{P}}$ is an order preserving choice function. Clearly the completeness of \mathbf{X} guarantees the existence of $x \in X$ such that $z = i(x)$.

$2 \Rightarrow 3$: Let \mathbf{X} be a ζ -closed subspace of $\Pi\beta_X$. Then since \mathbf{NA}_0 is complete, well-powered and cowell-powered, applying 37.6 in [15]: \mathbf{X} belongs to the epireflective hull of all discrete spaces. By Proposition 3.2 \mathbf{X} is \mathcal{V} -injective.

$3 \Rightarrow 1$: Suppose \mathbf{X} is \mathcal{V} -injective in \mathbf{NA}_0 . In view of Proposition 2.7 (2) we can conclude that X is ζ -closed in every Hausdorff non-Archimedean space in which it is embedded. In particular $i : \mathbf{X} \rightarrow \Pi\beta_X$ is a ζ -closed embedding. Now if f is any order preserving choice function, Proposition 3.5 implies that f converges to some point of X . \square

Let \mathbf{CNA}_0 be the full subconstruct of \mathbf{NA}_0 consisting of the complete Hausdorff non-Archimedean spaces. \mathbf{CNA}_0 is the unique \mathcal{V} -reflective subconstruct of \mathbf{NA} . In the next proposition we give an explicit description of the reflection, for a space \mathbf{X} in \mathbf{NA}_0 , which is in fact the "unique completion" of \mathbf{X} .

Theorem 3.7. *Let \mathbf{X} be a Hausdorff non-Archimedean space. Let \hat{X} be the set consisting of all order preserving choice functions of \mathbf{X} . For every $E \in \mathcal{E}_X$, let \mathcal{P} be the partition given by E . We define $\hat{E} = \{(f, g) \in \hat{X} \mid f(\mathcal{P}) = g(\mathcal{P})\}$. Consider the non-Archimedean space $\hat{\mathbf{X}}$ where the structure is given by the equivalence relations on \hat{X} that contain a relation of $\hat{\mathcal{E}} = \{\hat{E} \mid E \in \mathcal{E}_X\}$. We have the following:*

1. $\hat{\mathbf{X}}$ is a complete Hausdorff non-Archimedean space.
2. \mathbf{X} is a ζ -dense subspace of $\hat{\mathbf{X}}$.

Proof.

1. Suppose that $f, g \in \hat{X}$ are different. Then there is a $\mathcal{P} \in \beta_X$ such that $f(\mathcal{P}) \neq g(\mathcal{P})$. So $\hat{\mathbf{X}}$ is Hausdorff.

Let \hat{f} be an order preserving choice function on $\hat{\mathbf{X}}$. We make an order preserving choice function on \mathbf{X} as follows. Each $\mathcal{P} \in \beta_X$ has a corresponding equivalence relation $E_{\mathcal{P}}$, for which $\hat{E}_{\mathcal{P}}$ has a partition $\hat{\mathcal{P}}$ on \hat{X} . We define $f : \beta_X \rightarrow \cup \beta_X : \mathcal{P} \mapsto \hat{f}(\hat{\mathcal{P}})$. For any $\hat{\mathcal{P}}$ we have that $f \in \hat{f}(\hat{\mathcal{P}})$. Hence \hat{f} converges to f , so $\hat{\mathbf{X}}$ is complete.

2. Consider the following map:

$$j : \mathbf{X} \rightarrow \hat{\mathbf{X}} : x \mapsto (f_x : \beta_X \rightarrow \cup \beta_X : \mathcal{P} \mapsto [x]_{\mathcal{P}})$$

Clearly this f_x always is an order preserving choice function. Since \mathbf{X} is Hausdorff j obviously is injective.

Let $E \in \mathcal{E}_X$, clearly $E = (j \times j)^{-1}(\hat{E})$. Therefore j is initial.

Let $f \in \hat{X}$ and let $F_1, F_2 \in \mathcal{E}_{\hat{X}}$ which coincide on $j(X)$. There exist $E_1, E_2 \in \mathcal{E}_X$ and the corresponding partitions $\mathcal{P}_1, \mathcal{P}_2$ such that $\hat{E}_1 \subset F_1$ and $\hat{E}_2 \subset F_2$.

We have that $\hat{E}_1[f] = \{g \in \hat{X} \mid f(\mathcal{P}_1) = g(\mathcal{P}_1)\}$. Since $\emptyset \neq f(\mathcal{P}_1) \in \mathcal{P}_1$ there is an $x \in f(\mathcal{P}_1)$ such that $f_x \in \hat{E}_1[f]$, so $(f, f_x) \in \hat{E}_1 \subset F_1$. Analogously there is a $y \in f(\mathcal{P}_2)$ for which $(f, f_y) \in \hat{E}_2 \subset F_2$.

Since F_1, F_2 coincide on $j(X)$, we have that $(j \times j)^{-1}(F_1) = (j \times j)^{-1}(F_2)$, the latter corresponding to a $\mathcal{P}' \in \beta_X$ for which $\mathcal{P}_1, \mathcal{P}_2 \prec \mathcal{P}'$. Since f is order preserving and because of our choice of x and y we have that $x, y \in f(\mathcal{P}')$. Hence $(f_x, f_y) \in F_2$ and then also $(f, f_x) \in F_2$. Therefore $f_x \in F_1[f] \cap F_2[f] \cap j(X)$. So finally $f \in \zeta_{\hat{X}}(j(X))$.

□

From the previous theorem and starting with \mathbf{X} in \mathbf{NA}_0 we now can conclude that with respect to the class \mathcal{V} of ζ -dense embeddings, (\hat{X}, j) is the unique completion of \mathbf{X} . Indeed if $r_X : \mathbf{X} \rightarrow R\mathbf{X}$ is the reflection of \mathbf{X} in \mathbf{CNA}_0 , then

with the notations of Theorem 3.7 we can consider the diagram:

$$\begin{array}{ccc} & & RX \\ & \nearrow r_X & \\ X & \xrightarrow{j} & \hat{X} \simeq R\hat{X} \end{array}$$

Since \mathbf{CNA}_0 is firmly \mathcal{V} -reflective and j is ζ -dense, this means that $r(j)$ is an isomorphism.

From this it follows that a uniformly continuous map $u : \mathbf{X} \rightarrow \mathbf{Y}$ from a Hausdorff non-Archimedean space \mathbf{X} to a complete Hausdorff non-Archimedean space \mathbf{Y} can be uniquely extended to a uniformly continuous map $\hat{u} : \hat{\mathbf{X}} \rightarrow \mathbf{Y}$. We describe this extension explicitly as follows.

Let $f \in \hat{X}$. For any $\mathcal{P} \in \beta_Y$ we know that $u^{-1}(\mathcal{P}) \in \beta_X$, hence $f(u^{-1}(\mathcal{P}))$ is a class of $u^{-1}(\mathcal{P})$ and $u(f(u^{-1}(\mathcal{P})))$ is a subset of a class from \mathcal{P} , which we will write as $f_u(\mathcal{P})$. This defines an order preserving choice function as follows:

$$f_u : \beta_Y \rightarrow \cup \beta_Y : \mathcal{P} \mapsto f_u(\mathcal{P})$$

The extension of u is then given by:

$$\hat{u} : \hat{\mathbf{X}} \rightarrow \mathbf{Y} : f \mapsto \lim f_u$$

where $\lim f_u$ is the unique limit of f_u , which exists since \mathbf{Y} is Hausdorff and complete.

4 The case of non-Archimedean uniform spaces

In this last section we will now show that the previously described completion is in fact a generalization of the classical completion of a Hausdorff non-Archimedean uniform space, as described in [5],[21],[22].

Let \mathbf{X} be a Hausdorff non-Archimedean uniform space as introduced by Monna in [17], described by a collection of entourages \mathcal{U} . We write \mathcal{E} for the collection of equivalence relations in \mathcal{U} and β for the corresponding collection of partitions. Obviously \mathbf{X} uniquely corresponds to a Hausdorff non-Archimedean space in our sense.

Proposition 4.1. *Let \mathbf{X} be a non-Archimedean uniform space and let \mathcal{F} be a minimal Cauchy filter in \mathbf{X} . Then there is a unique element $(z_{\mathcal{P}})_{\mathcal{P} \in \beta} \in \Pi\beta$ such that $\mathcal{F} = \text{stack } \{z_{\mathcal{P}} | \mathcal{P} \in \beta\}$.*

Proof. We only have to check uniqueness since such a collection exists because \mathcal{F} is minimal. If $\mathcal{F} = \text{stack } \{z_{\mathcal{P}} | \mathcal{P} \in \beta\} = \text{stack } \{z'_{\mathcal{P}} | \mathcal{P} \in \beta\}$, then for every \mathcal{P} $z_{\mathcal{P}} \cap z'_{\mathcal{P}}$ is nonempty since \mathcal{F} is a filter. Hence $z_{\mathcal{P}} = z'_{\mathcal{P}}$. \square

Proposition 4.2. *Let \mathbf{X} be a non-Archimedean uniform space. Then there is a one to one correspondence between the order preserving choice functions of \mathbf{X} and the minimal Cauchy filters of \mathbf{X} .*

Proof. Let A be the set of all order preserving choice functions of \mathbf{X} and let B denote the set of all its minimal Cauchy filters. The following maps describe the needed one to one correspondence.

$$F : A \rightarrow B : f \mapsto \mathcal{F}_f$$

where $\mathcal{F}_f = \text{stack } \{z_{\mathcal{P}} | \mathcal{P} \in \beta_X\}$ with $z_{\mathcal{P}} = f(\mathcal{P})$ for $\mathcal{P} \in \beta_X$.

$$G : B \rightarrow A : \mathcal{F} \mapsto f_{\mathcal{F}}$$

where $f_{\mathcal{F}}(\mathcal{P}) = z_{\mathcal{P}}$ is uniquely defined by Proposition 4.1.

By definition \mathcal{F}_f contains arbitrarily small sets. For $\mathcal{P}, \mathcal{Q} \in \beta_X$ we have $z_{\mathcal{P} \wedge \mathcal{Q}} \subset z_{\mathcal{P}} \cap z_{\mathcal{Q}}$, so \mathcal{F}_f is a filter. By the same argument as in the proof of Proposition 4.1 one has that \mathcal{F}_f is minimal.

$f_{\mathcal{F}}$ is a well defined choice function by Proposition 4.1. For $\mathcal{P}, \mathcal{Q} \in \beta_X$ we have $z_{\mathcal{P} \wedge \mathcal{Q}} = z_{\mathcal{P}} \cap z_{\mathcal{Q}}$. It follows that $f_{\mathcal{F}}$ is order preserving.

After a simple verification one sees that F and G are bijective and inverse to one another. □

Since through the bijections F and G convergent order preserving choice functions correspond to convergent minimal Cauchy filters we can conclude the following.

Corollary 4.3. *The completion as developed in Theorem 3.7, when applied to a Hausdorff non-Archimedean uniform space reduces to the standard completion.*

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